PHYSICS LETTERS A

Physics Letters A 277 (2000) 13–17

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Pulse-enhanced stochastic resonance

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Received 15 August 2000; received in revised form 6 October 2000; accepted 6 October 2000
Communicated by C.R. Doering

Abstract

By adding constant-amplitude pulses to a noisy bistable system, we enhance its response to monochromatic signals, significantly magnifying its unpulsed stochastic resonance. We observe the enhancement in both numerical simulations and in analog electronic experiments. This simple noninvasive control technique should be especially useful in noisy bistable systems that are difficult or impossible to modify internally. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 05.40.-a; 05.45.-a; 02.50.-r; 87.80.+y

In the phenomenon of stochastic resonance (SR), a nonzero value of noise optimizes the response of a nonlinear system to a deterministic signal [1]. Over the past two decades, SR has generated much interest, and it has been demonstrated in numerous diverse experiments, involving physical, chemical, and biological systems [2]. Recently, Chow et al. [3] demonstrated how to enhance SR in a neuronal model by modulating the intensity of the input noise. Moreover, Gammaitoni et al. [4] were able to *control* SR, so as to either suppress or enhance the output power at the signal frequency, by sinusoidally modulating the barrier height between the two wells of a bistable system. Unfortunately, in many systems of interest, especially bioengineering applications involving neurons and neuronal arrays [5], it is difficult or impossible to modulate the relevant barrier. In this Letter, we enhance SR by simply adding *external* control pulses that increase the likelihood of switching between states, thereby obviating the need to internally modify the system.

Consider a noisy bistable oscillator evolving according to

$$m\ddot{x} + \gamma\dot{x} = -V'[x] + F_N[t], \tag{1}$$

where the accent denotes differentiation with respect position and the over-dots indicate differentiation with respect to time. The bistable potential defined by $V[x]/V_B = -2(x/R_B)^2 + (x/R_B)^4$ has a barrier of height $V_B = 256$, half width (or radius) $R_B = 5.66$, and maximum gradient (or maximum force) $F_M = 8V_B/\sqrt{27}\,R_B = 69.7$. The oscillator's internal (background) noise engenders a stochastic force $F_N[t] = \sigma N[t]$, where $N[t]$ represents band-limited Gaussian

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PII: S0375-9601(00)00665-4

white noise with zero mean and unit root-mean-square amplitude. Because bistable SR is traditionally studied in the regime where viscosity dominates inertia $\gamma\dot{x} \gg m\ddot{x}$, we simplify the analysis by setting $\gamma = 1$ and $m = 0$.

To enhance the response of the oscillator Eq. (1) to monochromatic inputs, we modify it by adding a controller that applies pulses $F_P[x] = -A_P x/|x|$ so that

$$m\ddot{x} + \gamma\dot{x} = -V'[x] + F_N[t] + F_P[x]$$
$$= -V'_{\text{eff}}[x] + F_N[t]. \tag{2}$$

Note that the pulses $F_P[x]$ depend *implicitly* but crucially on time, so that if the oscillator is on the left side of the barrier, the controller pushes it to the right $F_P[x < 0] = +A_P$, and if the oscillator is on the right side, the controller pushes it to the left $F_P[x > 0] = -A_P$. This effectively rocks the potential back and forth (nonperiodically) so as to encourage the oscillator to hop the central barrier. Indeed, the pulsed oscillator moves in an effective potential $V_{\text{eff}} = V - xF_P$ with a *lower* effective barrier height, as displayed in Fig. 1.

Finally, to the modified system Eq. (2), noisy oscillator plus controller, we add a monochromatic drive (or "signal") $F_D[t] = A_D \sin[2\pi f_D t]$, so that

$$m\ddot{x} + \gamma\dot{x} = -V'_{\text{eff}}[x] + F_N[t] + F_D[t]. \tag{3}$$

A weak drive amplitude $A_D = 0.11F_M = 8$ guarantees that the deterministic dynamics is subthreshold. (Because drive amplitudes of $A_D \geqslant F_M$ effectively rock the potential so that its inter-well barrier periodically disappears, the maximum force F_M is also known as the deterministic switching threshold.)

We numerically integrate the pulsed stochastic differential Eq. (3) using a first-order technique [6] with a time step $dt = 0.005$. We generate Gaussian noise using the Box–Muller algorithm [7] and a pseudorandom number generator. The finite time step slightly correlates the noise and band limits its spectrum to a Nyquist frequency $f_N = 1/2dt = 100$. Fig. 1 displays a sample time series with the accompanying pulses, along with the effective potential for reference.

We next estimate the mean square amplitude per frequency (or power spectrum) $S[f]$ of a long time series by averaging the spectra of many segments of the time series. Typically, we average 2^{10} spectra each

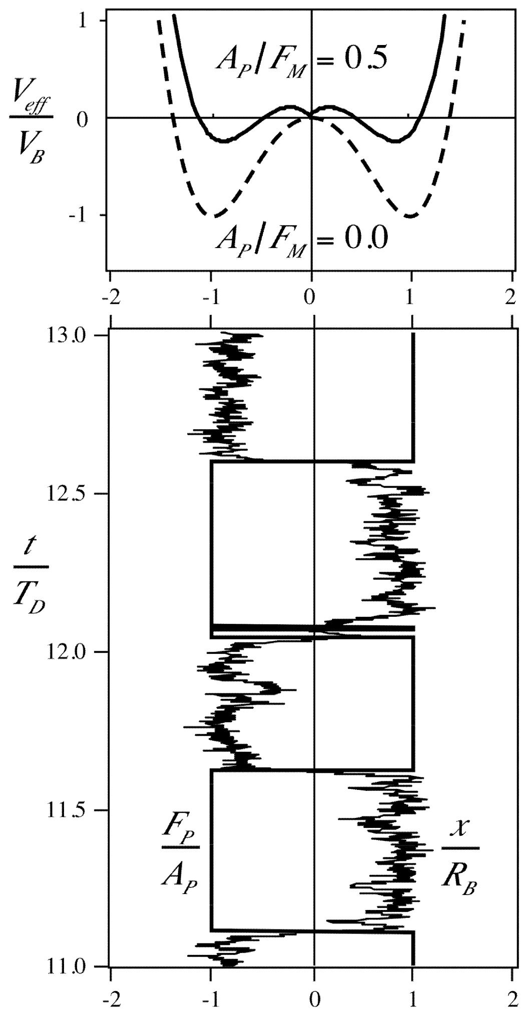

Fig. 1. Time series of a noisy, highly damped, bistable oscillator (jagged trace) subject to a non-periodic pulse controller (discrete trace). The effective potential V_{eff} has a reduced barrier height (as well as a stable cusp at the origin). The bistable potential is characterized by $V_B = 256$, $R_B = 5.66$, and hence, $F_M = 69.7$, while the drive parameters are $1/T_D = f_D = 0.195$ and $A_D = 0.11F_M = 8$.

containing 2^5 periods of the drive. Each spectrum consists of the magnitude squared of a normalized discrete Fourier transform [7]. We first filter the time series so as to remove intra-well oscillation and focus on inter-well hopping. Specifically, before applying a fast Fourier transform algorithm to the time series, we replace every negative x with -1 and every positive x with $+1$. Filtering simplifies the SNR curves (such as those of Fig. 2) by suppressing

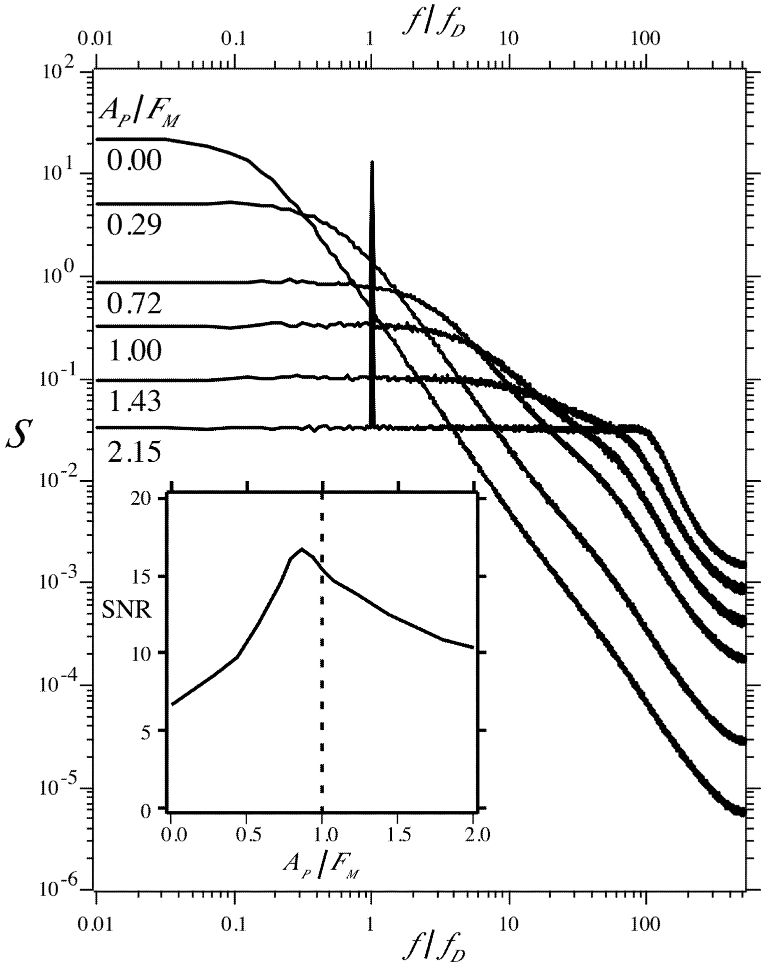

Fig. 2. Spectral densities S versus frequency f, for a series of pulse amplitudes A_P and fixed noise root-mean-square amplitude $\sigma = 2.15 F_M = 150$. Each spectrum consists of a sharp peak at the drive frequency f_D superimposed on a Lorentzian background. Inset: signal-to-noise ratio SNR versus pulse amplitude A_P, for fixed noise amplitude $\sigma = 2.15 F_M = 150$. A pulse amplitude $A_P \approx F_M$ optimizes the SNR.

large SNRs at low noise amplitudes, and thereby highlighting the local maxima at intermediate noise amplitudes. However, we observe pulse-enhanced SR with or without filtering [10].

From a spectrum, we estimate an output signal-to-noise ratio by $\mathrm{SNR} = 10 \log_{10}[S_D/\overline{S}_0]$, where $S_D = S[f_D]$ is the spectrum at the drive frequency and $\overline{S}_0$ is an estimate of the background spectrum near but not at the drive frequency. (The conventional factor of 10 expresses the result in decibels.) This traditional SNR definition [9] is appropriate because we want to quantify the response of the modified system Eq. (2), the pulsed (or controlled) oscillator, to a *monochromatic* drive $F_D[t]$.

Fig. 2 displays a series of spectra for different pulse amplitudes A_P at fixed noise amplitude $\sigma = 2.15 F_M = 150$. Each spectrum consists of a sharp peak at the drive frequency f_D superimposed on a Lorentzian background [8]. (The rises in the high-frequency tails of the spectra are unavoidable aliasing artifacts [7].) Increasing pulse amplitudes flatten the spectra while preserving their area, which is necessar-

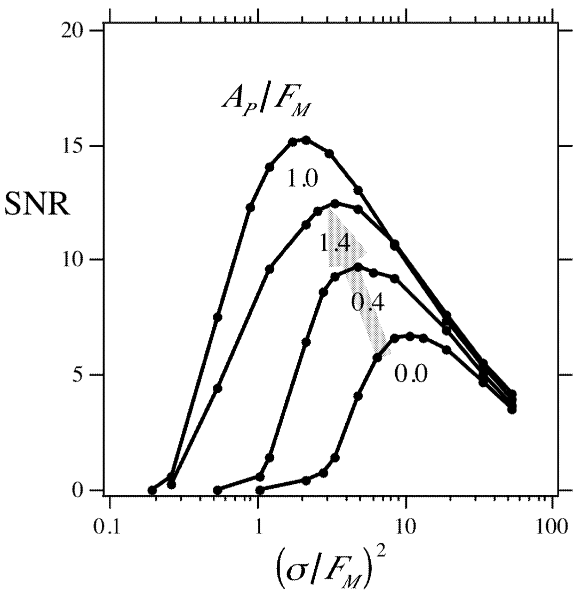

Fig. 3. SNR versus internal (background) noise mean square amplitude σ^2, for a series of pulse amplitudes A_P. Increasing the pulse amplitudes lowers the effective barrier heights and shifts the SR peaks to lower noise amplitudes and higher SNRs; gray arrow reflects a theoretical model. A pulse amplitude $A_P \approx F_M$ enhances the SR by about 10 dB.

ily the unit mean square amplitude of the filtered time series. Large pulses enable even weak noise to cause inter-well hopping, thereby flattening or "whitening" the resulting noisy time series.

The inset to Fig. 2 displays the corresponding SNRs. Small to moderate pulse amplitudes $A_P \leqslant F_M$ cooperate with the internal (background) noise and with the drive to increase the SNR, and a pulse amplitude comparable to the maximum force provided by the potential $A_P \approx F_M$ maximizes the SNR. Slightly larger pulse amplitudes $A_P \gtrsim F_M$ degrade the SNR by stimulating the oscillator to hop the inter-well barrier irrespective of the phase of the drive.

Fig. 3 displays SNR versus internal (background) noise mean square amplitude σ^2, for a series of pulse amplitudes A_P. For low noise, there is only intra-well motion, which the filter eliminates, and the SNR vanishes. (Although, the infinite "tails" of *ideal* Gaussian noise do induce rare barrier hopping.) For moderate noise amplitudes, the SNRs exhibit prominent local maxima, the signature of classical SR. Small to moderate pulse amplitudes $A_P \leqslant F_M$ cause the local maxima to drift to lower noise amplitudes and higher SNRs, culminating in a nearly 10 dB enhancement over the unpulsed SR. Very large pulse amplitudes $A_P \gg F_M$ destroy the SR by rendering the inter-well barrier insignificant and the potential effectively monostable, so that the SNRs decrease monotonically with noise.

We experimented with adding a hysteretic threshold to the pulses to reduce the "chatter" in their application (an example of which can be seen in Fig. 1 near $t = 12.1T_D$). Adding a small ($\sim 0.1R_B$) hysteretic threshold improved the SR enhancement slightly (by an additional $\sim$2 dB). However, larger thresholds did not yield further improvements. (We have also successfully tested a variety of other feedback schemes, such as negative proportional feedback [10].)

The essential mechanism of pulse-enhanced SR is the effective reduction in the height of the barrier separating the two wells of the bistable potential. Although the effective potential $V_{\text{eff}} = V - xF_P$ is stationary, it is not bistable (note the stable cusp at the origin in Fig. 1). Nevertheless, we can employ the McNamara–Wiesenfeld theory of SR [8] to estimate the shift and rise of the SR peak with increasing pulse amplitude and decreasing barrier height, where the effective barrier height is the difference between the local maximum and minimum potential energies. This theoretical result, which is indicated by the bold gray arrow in Fig. 3, is in good agreement with the simulations.

We have also observed pulse-enhanced SR experimentally in an analog electronic circuit. Specifically, we constructed a circuit [10] of passive elements (resistors, capacitors) and active elements (operational amplifiers), whose voltage as a function of time mimics the position of the driven pulsed oscillator described by Eq. (3). (The review by Gammaitoni et al. in Ref. [2] provides a good survey of such techniques.) Fig. 4 displays the experimental results, which are in good qualitative agreement with the simulations.

Pulse-enhanced SR is a simple strategy that an experimentalist can exploit to magnify a bistable stochastic resonance. Although the pulses are controlled by real-time monitoring of the time series, their structure $A_P \approx F_M$ depends only on the shape of the potential (and not at all on the frequency of the monochromatic drive), and hence may be determined before the experiment begins. Furthermore, this nonin-

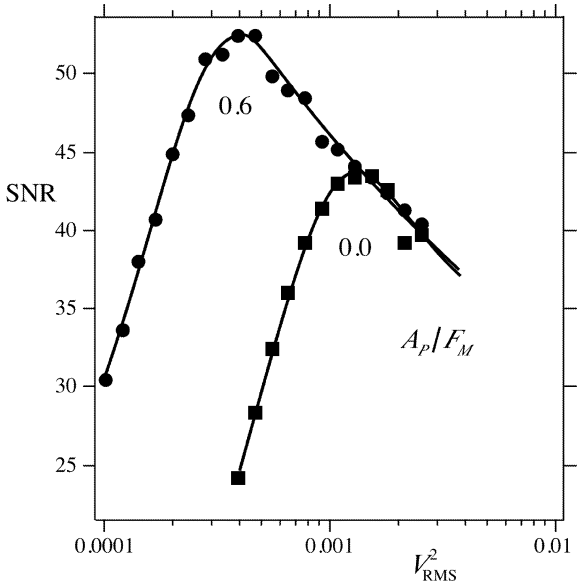

Fig. 4. Pulse-enhanced SR in an analog electronic circuit. A pulse amplitude $A_P = 0.6F_M$ shifts the circuit's SR peak to lower noise amplitudes and higher SNRs. The circuit's bistable potential is characterized by $V_B = 128$, $R_B = 5.66$, and hence $F_M = 34.8$, while its drive is determined by $f_D = 0.195$ and $A_D = 0.29F_M = 10$.

vasive technique requires only the application of *external* pulses, rather than the *internal* modification of the potential, even as it *effectively* depresses the interwell barrier.

Acknowledgements

J.F.L. thanks The College of Wooster for making possible his sabbatical at Georgia Tech. W.L.D. and M.L.S. acknowledge the Office of Naval Research, Physical Sciences Division for support. M.L.S. also acknowledges the NSWC ILIR Program. We thank Luca Gammaitoni for helpful discussions.

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